# Stanley depth of monomial ideals with small number of generators

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**Abstract**

For a monomial ideal $I \subset S = K[x_1,\dots,x_n]$, we show that $\operatorname{sdepth}(S/I) \geq n - g(I)$, where $g(I)$ is the number of the minimal monomial generators of $I$. If $I = vI'$, where $v \in S$ is a monomial, then we see that $\operatorname{sdepth}(S/I) = \operatorname{sdepth}(S/I')$. We prove that if $I$ is a monomial ideal $I \subset S$ minimally generated by three monomials, then $I$ and $S/I$ satisfy the Stanley conjecture. Given a saturated monomial ideal $I \subset K[x_1,x_2,x_3]$ we show that $\operatorname{sdepth}(I) = 2$. As a consequence, $\operatorname{sdepth}(I) \geq \operatorname{sdepth}(K[x_1,x_2,x_3]/I) + 1$ for any monomial ideal in $I \subset K[x_1,x_2,x_3]$.

**Keywords:** Stanley depth, monomial ideal.

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# Introduction

Let $K$ be a field and $S = K[x_1,\dots,x_n]$ the polynomial ring over $K$. Let $M$ be a $\mathbb{Z}^n$-graded $S$-module. A *Stanley decomposition* of $M$ is a direct sum $\mathcal{D} : M = \bigoplus_{i=1}^r m_i K[Z_i]$ as $K$-vector space, where $m_i \in M$, $Z_i \subset \{x_1,\dots,x_n\}$ such that $m_iK[Z_i]$ is a free $K[Z_i]$-module. We define $\operatorname{sdepth}(\mathcal{D}) = min_{i=1}^r|Z_i|$ and $\operatorname{sdepth}(M) = max\{\operatorname{sdepth}(M)|\ \mathcal{D}$ is a Stanley decomposition of $M\}$. The number $\operatorname{sdepth}(M)$ is called the *Stanley depth* of $M$. Herzog, Vladoiu and Zheng show in [8] that this invariant can be computed in a finite number of steps if $M = I/J$, where $J \subset I \subset S$ are monomial ideals.

There are two important particular cases. If $I \subset S$ is a monomial ideal, we are interested in computing $\operatorname{sdepth}(S/I)$ and $\operatorname{sdepth}(I)$. There are some papers regarding this problem, like [8],[12],[10], [14] and [5]. Stanley's conjecture says that $\operatorname{sdepth}(S/I) \geq \operatorname{depth}(S/I)$, or in the general case, $\operatorname{sdepth}(M) \geq \operatorname{depth}(M)$, where $M$ is a finitely generated multigraded $S$-module. The Stanley conjecture for $S/I$ was proved for $n \leq 5$ and in other special cases, but it remains open in the general case. See for instance, [4], [7], [9], [1], [3] and [11].

Let $I \subset S$ be a monomial ideal. We assume that $G(I) = \{v_1,\dots,v_m\}$, where $G(I)$ is the set of minimal monomial generators of $I$. We denote $g(I) = |G(I)|$, the number of minimal generators of $I$. Let $v = GCD(u|\ u \in G(I))$. It follows that $I = vI'$, where $I' = (I : v)$. For a monomial $u \in S$, we denote $supp(u) = \{x_i :\ x_i|u\}$. We denote $supp(I) = \{x_i :\ x_i|u$ for some $u \in G(I)\}$. We denote $c(I) = |supp(I')|$. In the first section, we prove results regarding some relations between $\operatorname{sdepth}(S/I)$, $\operatorname{sdepth}(I)$, $g(I)$ and $c(I)$.

In the second section, we give some applications. We prove that a monomial ideal $I \subset S$ minimally generated by three monomials has $\operatorname{sdepth}(I) = n-1$, see Theorem 2.4. We prove that if $I$ is a monomial ideal $I \subset S$ minimally generated by three monomials, then $I$ and $S/I$ satisfy the Stanley conjecture, see Theorem 2.6.

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In the third section, we prove that if $I \subset K[x_1, x_2, x_3]$ is saturated, then $\operatorname{sdepth}(I) \geq 2$, see Proposition 3.1. As a consequence, $\operatorname{sdepth}(I) \geq \operatorname{sdepth}(K[x_1, x_2, x_3]/I) + 1$ for any monomial ideal in $I \subset K[x_1, x_2, x_3]$, see Corollary 3.2, thereby giving in this special case an affirmative answer to a question raised by Rauf in [13].

# 1 Preliminary results

We recall the following result of Herzog, Vladoiu and Zheng.

**Proposition 1.1.** *([8, Proposition 3.4]) Let $I \subset S$ be a monomial ideal. Then:*

$$\operatorname{sdepth}(I) \geq max\{1, n - g(I) + 1\}.$$

In the following, we give a similar result:

**Proposition 1.2.** *Let $I \subset S$ be a monomial ideal. Then* $\operatorname{sdepth}(S/I) \geq n - g(I)$.

*Proof.* In order to prove, we use a strategy similar with the Janet's algorithm, see [2]. We use induction on $n \geq 1$. If $n = 1$ there is nothing to prove. Denote $m = g(I)$. If $m = 1$, $I$ is principal and thus $\operatorname{sdepth}(S/I) = n - 1$. Suppose $n > 1$ and $m > 1$. Let $q = deg_{x_n}(I) := \max\{j : \ x_n^j | u$ for some $u \in G(I)\}$. For all $j \leq q$, we denote $I_j$ the monomial ideal in $S' = K[x_1, \ldots, x_{n-1}]$ such that $I \cap x_n^j S' = x_n^j I_j$. Note that $g(I_j) < g(I)$ for all $j < q$ and $g(I_q) \leq g(I)$. We have

$$S/I = S'/I_0 \oplus x_n(S'/I_1) \oplus \cdots \oplus x_n^{q-1}(S'/I_{q-1}) \oplus x_n^q(S'/I_q)[x_n].$$

It follows that $\operatorname{sdepth}(S/I) \geq \min\{\operatorname{sdepth}(S'/I_j), j < q, \operatorname{sdepth}(S'/I_q) + 1\}$. By induction hypothesis, it follows that $\operatorname{sdepth}(S'/I_j) \geq n - 1 - q(I_j) \geq n - 1 - (m - 1) = n - m$ for all $j < q$. Also, $\operatorname{sdepth}(S'/I_q) \geq n - 1 - g(I_q) \geq n - 1 - m$. This completes the proof. □

For any monomial ideal $J \subset S$, we denote $J^c$ the $K$-vector space spanned by all the monomials not contained in $J$. With this notation, we have the following lemma.

**Lemma 1.3.** *Let $I \subset S = K[x_1, \ldots, x_n]$ be a monomial ideal and $v \in S$ a monomial. Then $I = ((v)^c \cap I) \oplus v(I : v)$ and $I^c = ((v)^c \cap I^c) \oplus v(I : v)^c$.*

*Proof.* We have $I = S \cap I = ((v)^c \oplus (v)) \cap I = ((v)^c \cap I) \oplus ((v) \cap I)$. In order to complete the proof, it is enough to show that $((v) \cap I) = v(I : v)$. Indeed, if $w \in (v) \cap I$ is a monomial, then $w = vy$ for some monomial $y \in S$. Moreover, since $vy = w \in I$ it follows that $y \in (I : v)$ and thus $w \in v(I : v)$. The inclusion $((v) \cap I) \supseteq v(I : v)$ is similar. Analogously, we prove the second statement. □

**Theorem 1.4.** *Let $I \subset S$ be a monomial ideal which is not principal. Assume $I = vI'$, where $v \in S$ is a monomial and $I' = (I : v)$. Then:*

*(1)* $\operatorname{sdepth}(S/I) = \operatorname{sdepth}(S/I')$.

*(2)* $\operatorname{sdepth}(I) = \operatorname{sdepth}(I')$.

*Proof.* (1) By Lemma 1.3, $S/I = I^c = (v)^c \oplus v(I'^c)$. Given a Stanley decomposition $S/I' = \bigoplus_{i=1}^r u_i' K[Z_i]$ of $S/I'$, it follows that $\bigoplus_{i=1}^r v u_i' K[Z_i]$ is a Stanley decomposition of $v(I'^c)$. On the other hand, since $\operatorname{sdepth}(S/(v)) = 1$, one can easily give a Stanley decomposition $\mathcal{D}$ of $(v)^c$ with $\operatorname{sdepth}(\mathcal{D}) = n-1$. Thus, we obtain a Stanley decomposition of $S/I$ with its Stanley depth $\geq \operatorname{sdepth}(S/I')$. It follows that $\operatorname{sdepth}(S/I) \geq \operatorname{sdepth}(S/I')$.

In order to prove the converse inequality, we consider $S/I = \bigoplus_{i=1}^r u_i K[Z_i]$ a Stanley decomposition of $S/I$. It follows that $v(I'^c) = \bigoplus_{i=1}^r (u_i K[Z_i] \cap v(I'^c)) = \bigoplus_{i=1}^r (u_i K[Z_i] \cap (v))$. The last equality follows from the fact that $I^c \cap (v) = v(I'^c)$. We claim that $u_i K[Z_i] \cap (v) \neq (0)$ implies $LCM(u_i, v) \in u_i K[Z_i]$. Indeed, if $LCM(u_i, v) \notin u_i K[Z_i]$ it follows that $v/GCD(u_i, v) \notin K[Z_i]$ and therefore, there exists $x_j | v/GCD(u_i, v)$ such that $x_j \notin Z_i$. Thus, $v$ cannot divide any monomial of the form $u_i y$, where $y \in K[Z_i]$ and therefore $u_i K[Z_i] \cap (v) = (0)$, a contradiction. Now, since $LCM(u_i, v) \in u_i K[Z_i]$, it follows that $LCM(u_i, v) K[Z_i] \subset u_i K[Z_i]$. Obviously, $LCM(u_i, v) K[Z_i] \subset (v)$ and thus $LCM(u_i, v) K[Z_i] \subset u_i K[Z_i] \cap (v)$. On the other hand, if $u \in u_i K[Z_i] \cap (v)$ is a monomial, it follows that $u_i | u$ and $v | u$ and therefore, $LCM(u_i, v) | u$. Since $u \in u_i K[Z_i]$, it follows that $u = u_i w_i$, where $supp(w_i) \subset Z_i$. Moreover, $supp(u/LCM(u_i, v)) \subset Z_i$ and thus, $u \in LCM(u_i, v) K[Z_i]$. We obtain $LCM(u_i, v) K[Z_i] = u_i K[Z_i] \cap (v)$. In conclusion,

$$vI'^c = \bigoplus_{(v) \cap u_i K[Z_i] \neq 0} LCM(u_i, v) K[Z_i] \text{ so } I'^c = \bigoplus_{(v) \cap u_i K[Z_i] \neq 0} \frac{u_i}{GCD(u_i, v)} K[Z_i].$$

It follows that $\operatorname{sdepth}(S/I') \geq \operatorname{sdepth}(S/I)$, as required.

(2) Follows from the linear space isomorphism $I' \cong vI' = I$. □

**Proposition 1.5.** *Let $I \subset S$ be a monomial ideal. Then:*

*(1)* $\operatorname{sdepth}(S/I) \geq n - c(I)$

*(2)* $\operatorname{sdepth}(I) \geq n - c(I) + 1$.

*Proof.* (1) Let $v = GCD(u | \ u \in G(I))$ and $I' = (I : v)$. By 1.4(1) we can assume that $I' = I$. By reordering the variables, we can assume that $I \subset (x_1, x_2, \ldots, x_m)$, where $m = c(I)$. We write $I = (I \cap K[x_1, \ldots, x_m])S$. [8, Lemma 3.6] implies $\operatorname{sdepth}(S/I) = \operatorname{sdepth}(K[x_1, \ldots, x_m]/(I \cap K[x_1, \ldots, x_m])) + n - m \geq n - m$. (2) The proof is similar, if we see that $\operatorname{sdepth}(I \cap K[x_1, \ldots, x_m]) \geq 1$, see 1.1. □

**Proposition 1.6.** *Let $I \subset S$ be a monomial ideal which is not principal with $c(I) = 2$ or $g(I) = 2$. Then* $\operatorname{sdepth}(I) = n - 1$ *and* $\operatorname{sdepth}(S/I) = n - 2$.

*Proof.* If $c(I) = 2$, then, by 1.5(2), it follows that $\operatorname{sdepth}(I) \geq n - c(I) + 1 = n - 1$. If $g(I) = 2$, by 1.1, it follows that $\operatorname{sdepth}(I) \geq n - 1$. On the other hand, $\operatorname{sdepth}(I) < n$, otherwise, $I$ would be principal. Thus $\operatorname{sdepth}(I) = n - 1$.

According to 1.5(1) or 1.2, $\operatorname{sdepth}(S/I) \geq n-2$ if $c(I)=2$ or, respectively, $g(I)=2$. We consider the case $c(I)=2$. Let $v = GCD(u|\ u \in G(I))$ and $I' = (I : v)$. By 1.4(1), we can assume that $I = I'$ and $supp(I) = \{x_1, x_2\}$. Since $v = 1$, it follows that $x_1^a, x_2^b \in G(I)$ for some positive integers $a$ and $b$. Therefore, $\operatorname{sdepth}(I \cap K[x_1, x_2]) = 0$ and moreover, [8, Lemma 3.6] implies $\operatorname{sdepth}(S/I) = \operatorname{sdepth}(K[x_1,x_2]/(I \cap K[x_1,x_2])) + n - 2 = n-2$.

We consider now the case $g(I) = 2$. Suppose $I = (u_1, u_2)$. By 1.4(1), we can assume $GCD(u_1, u_2) = 1$. Therefore, $I$ is a complete intersection and by [7, Proposition 1.2] or [12, Corollary 1.4], it follows that $\operatorname{sdepth}(S/I) = n-2$. $\square$

# 2 Stanley depth of monomial ideals with small number of generators

The following result is a particular case of [6, Theorem 1.4].

**Theorem 2.1.** *Let $I \subset S$ be a monomial ideal. Then* $\operatorname{sdepth}(S/I) = 0$ *if and only if* $depth(S/I) = 0$.

Note that $depth(S/I) = 0$ if and only if $I \neq I^{sat}$, where $I^{sat} = \bigcup_{k\geq 1}(I : (x_1, \ldots, x_n)^k)$ is the *saturation* of $I$.

**Corollary 2.2.** *Let $I \subset S$ be a monomial ideal. Then* $\operatorname{sdepth}(S/I) = 0$ *if and only if* $\operatorname{sdepth}(S/I^k) = 0$*, where $k \geq 1$.*

*Proof.* It is enough to notice that $\operatorname{depth}(S/I) = 0$ if and only if $\operatorname{depth}(S/I^k) = 0$, where $k \geq 1$, than we apply Theorem 1.1. $\square$

**Corollary 2.3.** *Let $I \subset S$ be a monomial ideal with $c(I) = n$ and $(x_1, \ldots, x_{n-1}) \subset \sqrt{I}$. Then* $\operatorname{sdepth}(S/I) = 0$.

*Proof.* Since $(x_1, \ldots, x_{n-1}) \subset \sqrt{I}$ it follows that for all $1 \leq j \leq n-1$, there exists a positive integer $a_j$ such that $x_j^{a_j} \in I$. Since $c(I) = n$ it follows that there exists a monomial $u \in G(I)$ with $x_n|u$. If $u = x_n^{a_n}$, it follows that $I$ is artinian and thus, by Theorem 2.1, $\operatorname{sdepth}(S/I) = 0$. Suppose this is not the case. We consider $w = u/x_n$. Obviously, $x_j^{a_j}w \in I$ for any $1 \leq j \leq n$, where $a_n := 1$. Thus, $w \in I^{sat} \setminus I$ and then $\operatorname{sdepth}(S/I) = 0$ by 2.1. $\square$

**Theorem 2.4.** *Let $I \subset S$ be a monomial ideal with $g(I) = 3$. Then* $\operatorname{sdepth}(I) = n-1$.

*Proof.* Denote $G(I) = \{v_1, v_2, v_3\}$. By 1.4(2), we can assume that $GCD(v_1, v_2, v_3) = 1$. If $I$ is a complete intersection, by [8, Proposition 3.8] or [14, Theorem 2.4], it follows that $\operatorname{sdepth}(I) = n-1$. If this is not the case, it follows that there exists a variable, let's say $x_n$, such that $x_n|v_1$,$x_n|v_2$ and $x_n$ does not divide $v_3$. Let $a = deg_{x_n}(v_1)$ and $b = deg_{x_n}(v_2)$ and suppose $a \leq b$. We have the following decomposition given by Lemma 1.3:

$$I = (I \cap (x_n^a)^c) \oplus x_n^a(I : x_n^a) = \bigoplus_{j=0}^{a-1} x_n^j v_3 K[x_1, \ldots, x_{n-1}] \oplus x_n^a(I : x_n^a).$$

Note that $g((I : x_n^a)) \leq g(I)$. If $g((I : x_n^a)) < 3$, we can find a Stanley decomposition for $(I : x_n^a)$ with it's Stanley depth $\geq n-1$ and we stop. Otherwise, we replace $I$ with $(I : x_n^a)$ and we repeat the previous procedure until we obtain an ideal with $\leq 2$ generators of a monomial complete intersection ideal. Finally, we obtain a Stanley decomposition of $I$ with it's Stanley depth equal to $n-1$. On the other hand, $\text{sdepth}(I) < n$, since $I$ is not principal. $\square$

**Example 2.5.** *Let $I = (x_1^3, x_2^2x_3^2, x_1x_2^3x_3)$. We have:*

$$I = ((x_2^2)^c \cap I) \oplus x_2^2(I : x_2^2) = x_1^3K[x_1, x_3] \oplus x_1^3x_2K[x_1, x_3] \oplus x_2^2(x_1^3, x_3^2, x_1x_2x_3),$$

*On the other hand,*

$$(x_1^3, x_3^2, x_1x_2x_3) = x_3^2K[x_2, x_3] \oplus x_1(x_1^2, x_3^2, x_2x_3) = x_3^2K[x_2, x_3] \oplus x_1^3K[x_1, x_2] \oplus x_1x_3(x_1^2, x_2, x_3).$$

*We obtain the following Stanley decomposition of $I$:*

$$I = x_1^3K[x_1, x_3] \oplus x_1^3x_2K[x_1, x_3] \oplus x_2^2x_3^2K[x_2, x_3] \oplus x_2^2x_1^3K[x_1, x_2] \oplus$$

$$\oplus x_1x_2^2x_3(x_2K[x_2, x_3] \oplus x_1x_2K[x_2, x_3] \oplus x_1^2K[x_1, x_2] \oplus x_3K[x_1, x_3] \oplus x_1^2x_2x_3K[x_1, x_2, x_3]).$$

**Theorem 2.6.** *Let $I \subset S$ be a monomial ideal with $g(I) \leq 3$. Then, $I$ and $S/I$ satisfy the Stanley's conjecture.*

*Proof.* It is well known that $\text{depth}(I) = \text{depth}(S/I) + 1$ and $\text{depth}(S/I) = n-1$ if and only if $I$ is principal. Thus, if $g(I) = 1$, there is nothing to prove. If $g(I) = 2$, by 1.6, we have $\text{sdepth}(S/I) = n-2$ and $\text{sdepth}(I) = n-1$. Since $I$ is not principal, it follows that $\text{depth}(S/I) \leq n-2$ and $\text{depth}(I) \leq n-1$. So we are done.

We consider the case $g(I) = 3$. By 2.4, $\text{sdepth}(I) = n-1$ and thus $\text{sdepth}(I) \geq \text{depth}(I)$. According to 1.2, $\text{sdepth}(S/I) \geq n-3$. Thus, if $\text{depth}(S/I) \leq n-3$ we are done.

Now, assume $\text{depth}(S/I) = n-2$. Denote $G(I) = \{v_1, v_2, v_3\}$, $v = GCD(v_1, v_2, v_3)$ and $I' = (I : v)$. By 1.4(1), $\text{sdepth}(S/I) = \text{sdepth}(S/I')$. On the other hand, by [13, Corollary 1.3], $\text{depth}(S/I')) \geq \text{depth}(S/I)$. In fact, $\text{depth}(S/I') = n-2$, since $I'$ is not principal. Thus, we can assume $I = I'$. Note that $I$ is not a complete intersection, otherwise $\text{depth}(S/I) = n-3$. Therefore, there exists a variable, let's say $x_n$, such that $x_n|v_1$,$x_n|v_2$ and $x_n$ does not divide $v_3$. Let $a = deg_{x_n}(v_1)$ and $b = deg_{x_n}(v_2)$ and suppose $a \leq b$. We have the following decomposition given by Lemma 1.3:

$$S/I = I^c = (I^c \cap (x_n^a)^c) \oplus x_n^a((I : x_n^a)^c) = \bigoplus_{j=0}^{a-1} x_n^j(S'/v_3S') \oplus x_n^a((I : x_n^a)^c),$$

where $S' = K[x_1, \ldots, x_{n-1}]$. Note that $\text{sdepth}_{S'} S'/v_3S' = (n-1)-1 = n-2$. We have $g((I : x_n^a)) \leq g(I)$. If $g((I : x_n^a)) < 3$, we can find a Stanley decomposition for $S/(I : x_n^a)$ with it's Stanley depth $>= n-2$ and we stop. Assume $g((I : x_n^a)) = 3$. By [13, Corollary 1.3], we have $\text{depth}(S/(I : x_n^a)) \geq \text{depth}(S/I) = n-2$. Thus, $\text{depth}(S/(I : x_n^a)) = n-2$ and therefore $(I : x_n^a)$ is not a complete intersection. We replace $I$ with $(I : x_n^a)$ and we continue the previous procedure. Finally, we obtain a Stanley decomposition for $S/I$ with it's Stanley depth equal to $n-2$. Therefore $\text{sdepth}(S/I) \geq n-2$, as required. $\square$

## 3 Monomial ideals in three variables

**Lemma 3.1.** *Let $I \subset S := K[x_1, x_2, x_3]$ be a monomial ideal with $v := GCD(u|u \in G(I)) = 1$. For $1 \le j \le 3$, we denote $S_j := K[Z_j]$ and $I_j = I \cap S_j$, where $Z_j = \{x_1, x_2, x_3\} \setminus \{x_j\}$. If $I^{sat} = I$ then there exists some $1 \le j \le 3$ such that $I_j^{sat} = I_j$.*

*Proof.* If $I = S$ there is nothing to prove, so we can assume $I \neq S$. Since $I^{sat} = I$, it follows that $\mathbf{m} = (x_1, x_2, x_3) \notin Ass(S/I)$. Since $v = 1$, it follows that $(x_j) \notin Ass(S/I)$ for all $1 \le j \le 3$. We denote $\mathbf{m}_j$ the ideal generated by the variables from $Z_j$. We have $Ass(S/I) \subset \{\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3\}$.

Thus, we can find a decomposition $I = \bigcap_{j=1}^{3} Q_j$ such that $Q_j$ is $\mathbf{m}_j$-primary or $Q_j = S$ for all $1 \le j \le 3$. It follows that $I_k = \bigcap_{j=1}^{3} (Q_j \cap S_k)$. We assume $Q_1 = (x_2^a, x_3^b, \ldots)$, $Q_2 = (x_1^c, x_3^d, \ldots)$ and $Q_3 = (x_1^e, x_2^f, \ldots)$ where $a, b, c, d, e, f$ are some nonnegative integers. By reordering $Q_i$'s, we can assume $a \ge f$. It follows that $Q_1 \cap S_3 = (x_2^a) \subset Q_3$. Therefore, $I_3 = (x_2^a) \cap (x_1^c) = (x_2^a x_1^c)$ is principal and thus, $I_3 = I_3^{sat}$. $\square$

**Proposition 3.2.** *Let $I \subset S := K[x_1, x_2, x_3]$ be a monomial ideal which is not principal. If $I = I^{sat}$ then* $\operatorname{sdepth}(I) = 2$.

*Proof.* We denote $v = GCD(u|u \in G(I))$ and $I' = (I : v)$. By Theorem 1.4(2), we have $\operatorname{sdepth}(I) = \operatorname{sdepth}(I')$. Since, also, $I^{sat} = vI'^{sat}$, we can assume $I = I'$. If $c(I) = 2$ or $g(I) = 2$, by Proposition 1.6, it follows that $\operatorname{sdepth}(I) = 2$. Now, we consider the case: $c(I) = 3$ and $g(I) \ge 3$. In the notations of Lemma 3.1, we can assume that $I_1^{sat} = I_1$ and $I_1$ is principal. Thus $\operatorname{sdepth}(I_1) = 2$. We write $I = I_1 \oplus x_1(I : x_1)$. Obviously, $I \subsetneq (I : x_1)$ and $(I : x_1)^{sat} = (I : x_1)$. We can use the same procedure for $(I : x_1)$. Finally, we obtain a Stanley decomposition of $I$ with its Stanley depth equal to 2. $\square$

**Corollary 3.3.** *If $I \subset K[x_1, x_2, x_3]$ is a monomial ideal, then* $\operatorname{sdepth}(I) \ge \operatorname{sdepth}(S/I) + 1$. *In particular, if* $\operatorname{sdepth}(I) = 1$*, then* $\operatorname{depth}(I) = 1$.

*Proof.* If $\operatorname{sdepth}(S/I) = 0$ there is nothing to prove, since $\operatorname{sdepth}(I) \ge 1$. If $\operatorname{sdepth}(S/I) = 1$, by Theorem 2.1, it follows that $I = I^{sat}$ and, by 3.1, $\operatorname{sdepth}(I) = 2$. On the other hand, $\operatorname{sdepth}(S/I) = 2$ if and only if $I$ is principal and, thus, if and only if $\operatorname{sdepth}(I) = 3$.

For the second statement, assume $\operatorname{depth}(I) > 1$, i.e. $\operatorname{depth}(S/I) > 0$. It follows by 2.1 that $I = I^{sat}$ and thus, by 3.2, $\operatorname{sdepth}(I) \ge 2$, a contradiction. $\square$

**Remark 3.4.** A similar result to Lemma 3.1 is not true for $n \ge 4$. Let $S = K[x_1, x_2, x_3, x_4]$, $Q_1 = (x_2^3, x_3^2, x_4)$, $Q_2 = (x_1^3, x_3, x_4^2)$, $Q_3 = (x_1^2, x_2, x_4^3)$, $Q_4 = (x_1, x_2^2, x_3^3)$ and $I = Q_1 \cap Q_2 \cap Q_3 \cap Q_4$. Let $Z_k = \{x_1, x_2, x_3, x_4\} \setminus \{x_k\}$ and $S_k := K[Z_k]$, where $1 \le k \le 4$. One can easily see that $I_k = I \cap S_k = \bigcap_{j=1}^{4} (Q_j \cap S_k)$ is a reduced primary decomposition of $I_k$. In particular, $\mathbf{m}_k = \sqrt{Q_k \cap S_k} \in Ass(S_k/I_k)$ and thus $I_k^{sat} \neq I_k$. On the other hand, $I = I^{sat}$.

# References


[1] Sarfraz Ahmad, Dorin Popescu "Sequentially Cohen-Macaulay monomial ideals of embedding dimension four", Bull. Math. Soc. Sc. Math. Roumanie 50(98), no.2 (2007), p.99-110.

[2] Imran Anwar "Janet's algorithm", Bull. Math. Soc. Sc. Math. Roumanie 51(99), no.1 (2008), p.11-19.

[3] Imran Anwar, Dorin Popescu "Stanley Conjecture in small embedding dimension", Journal of Algebra 318 (2007), p.1027-1031.

[4] J.Apel "On a conjecture of R.P.Stanley", Journal of Algebraic Combinatorics, 17(2003), p.36-59.

[5] Mircea Cimpoeas "Stanley depth for monomial complete intersection", Bull. Math. Soc. Sc. Math. Roumanie 51(99), no.3 (2008), p.205-211.

[6] Mircea Cimpoeas "Some remarks on the Stanley depth for multigraded modules", Le Mathematiche, Vol. LXIII (2008) Fasc. II, pp. 165171.

[7] Jürgen Herzog, Ali Soleyman Jahan, Siamak Yassemi "Stanley decompositions and partitionable simplicial complexes", Journal of Algebraic Combinatorics 27(2008), p.113-125.

[8] Jürgen Herzog, Marius Vladoiu, Xinxian Zheng "How to compute the Stanley depth of a monomial ideal", Journal of Algebra 2009, doi:10:1016/j.jalgebra.2008.01.006 in press

[9] Ali Soleyman Jahan "Prime filtrations of monomial ideals and polarizations", Journal of Algebra 312 (2007), p.1011-1032.

[10] Sumiya Nasir "Stanley decompositions and localization", Bull. Math. Soc. Sc. Math. Roumanie 51(99), no.2 (2008), p.151-158.

[11] Dorin Popescu "Stanley depth of multigraded modules", Journal of Algebra, 321 (10), 2009, p.2782-2797.

[12] Asia Rauf "Stanley decompositions, pretty clean filtrations and reductions modulo regular elements", Bull. Soc. Sc. Math. Roumanie 50(98), no.4 (2007), p.347-354.

[13] Asia Rauf "Depth and Stanley depth of multigraded modules", Preprint 2009 , to appear in Communications in Algebra.

[14] Yihuang Shen "Stanley depth of complete intersection monomial ideals and upper-discrete partitions", Journal of Algebra 321(2009), 1285-1292.



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